SOME SOLUTIONS OF THE STEFAN PROBLEM

A. M. Tsybin

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Some particular solutions of the Stefan problem are presented.

We consider the following problem:

$$\frac{\partial T}{\partial \tau} = a\frac{\partial^2 T}{\partial x^2} \quad (0<x<\xi(\tau),\ \ \tau>0), \tag{1}$$

$$T(0,\ \tau) = \varphi(\tau) < 0, \tag{2}$$

$$T(\xi(\tau),\ \tau) = 0, \tag{3}$$

$$\frac{\partial T}{\partial x}(\xi(\tau),\ \tau) = B\frac{d\xi}{d\tau}, \tag{4}$$

$$\xi(0) = 0. \tag{5}$$

We make the substitution $x = \xi(\tau) z$. Equation (1) then transforms to

$$\frac{\partial T}{\partial \tau} = \frac{\xi'(\tau)}{\xi(\tau)} z \frac{\partial T}{\partial z} + \frac{a}{\xi^2(\tau)} \cdot \frac{\partial^2 T}{\partial z^2}. \tag{6}$$

The variables in Eq. (6) are separable when and only when

$$\xi(\tau) = \beta\sqrt{\tau + \text{const}}, \quad \text{where } \beta = \text{const} > 0.$$

Of the Stefan problems in which separation of variables is effective, one can note the results obtained by Sanders [1]. Considering Eq. (5),

$$\xi(\tau) = \beta\sqrt{\tau}. \tag{7}$$

By substituting Eq. (7) into Eq. (6), performing the separation of variables, and allowing for Eq. (2), we obtain

$$T(x,\ \tau) = \sum_{(\lambda)} \tau^{-\lambda}\left[C_{1\lambda}\frac{x}{\xi(\tau)}\,\Phi\left(\lambda + \frac{1}{2},\ \frac{3}{2};\ -\frac{\beta^2 x^2}{4a\xi^2(\tau)}\right) + C_{2\lambda}\,\Phi\left(\lambda,\ \frac{1}{2};\ -\frac{\beta^2 x^2}{4a\xi^2(\tau)}\right)\right], \tag{8}$$

where $\Phi(\alpha, \gamma; z)$ is a degenerate hypergeometric function [3]; λ are numbers which, in general, are complex; $C_{1\lambda}$, and $C_{2\lambda}$ are real constants.

By fulfilling the conditions (2)-(4), we obtain

$$\sum_{(\lambda)} \tau^{-\lambda} C_{2\lambda} = \varphi(\tau), \tag{9}$$

$$\sum_{(\lambda)} \tau^{-\lambda}\left[C_{1\lambda}\,\Phi\left(\lambda + \frac{1}{2},\ \frac{3}{2};\ -\frac{\beta^2}{4a}\right) + C_{2\lambda}\,\Phi\left(\lambda,\ \frac{1}{2};\ -\frac{\beta^2}{4a}\right)\right] = 0, \tag{10}$$

$$\sum_{(\lambda)} \tau^{-\lambda}\left\{C_{1\lambda}\left[\frac{a}{\beta^2}\,\Phi\left(\lambda + \frac{3}{2},\ \frac{3}{2};\ -\frac{\beta^2}{4a}\right) - \frac{\lambda}{3}\,\Phi\left(\lambda + \frac{3}{2},\ \frac{5}{2};\ -\frac{\beta^2}{4a}\right)\right] - \lambda C_{2\lambda}\,\Phi\left(\lambda + 1,\ \frac{3}{2};\ -\frac{\beta^2}{4a}\right)\right\} = \frac{Ba}{2}. \tag{11}$$

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TABLE 1. Values of $f(\mu; \beta^2/4a)$ (20)

μ	$\beta^2/4a$ 0,01	0,1	0,5	1	3
—50	—75,2967	—50,1187	—50,0000	—50,0000	—50,0000
—45	—71,3062	—45,1606	—45,0000	—45,0000	—45,0000
—40	—67,3589	—40,2192	—40,0000	—40,0000	—40,0000
—35	—62,4565	—35,3018	—35,0001	—35,0000	—35,0000
—30	—59,6011	—30,4196	—30,0002	—30,0000	—30,0000
—25	—55,7947	—25,5895	—25,0005	—25,0000	—25,0000
—20	—52,0395	—20,8374	—20,0017	—20,0000	—20,0000
—15	—48,3375	—16,2032	—15,0061	—15,0001	—15,0000
—10	—44,6911	—11,7494	—10,0249	—10,0009	—10,0000
—9	—43,9688	—10,8881	—9,0338	—9,0014	—9,0000
—8	—43,2487	—10,0386	—8,0461	—8,0024	—8,0000
—7	—42,5310	—9,2022	—7,0636	—7,0041	—7,0000
—6	—41,8157	—8,3799	—6,0886	—6,0072	—6,0000
—5	—41,1028	—7,5731	—5,1248	—5,0013	—5,0000
—4	—40,3923	—6,7831	—4,1776	—4,0241	—4,0001
—3	—39,6842	—6,0114	—3,2557	—3,0464	—3,0004
—2	—38,9786	—5,2598	—2,3725	—2,0932	—2,0022
—1	—38,2755	—4,5299	—1,5490	—1,1965	—1,0143
0	—37,5749	—3,8236	—0,8173	—0,4330	—0,1371

The rule for the differentiation of a degenerate hypergeometric function [3] was used in the derivation of Eq. (11). If

$$C_{2\lambda} = -C_{1\lambda}\frac{\Phi\left(\lambda+\frac{1}{2},\ \frac{3}{2};\ -\frac{\beta^2}{4a}\right)}{\Phi\left(\lambda,\ \frac{1}{2};\ -\frac{\beta^2}{4a}\right)}, \tag{12}$$

the condition (10) is satisfied.

Let

$$\varphi(\tau) = P + Q\tau^{-\mu}, \tag{13}$$

where P and Q are real negative numbers. By successively setting $\lambda = 0$ and $\lambda = \mu$ and taking [3] into consideration, we then find

$$P + Q\tau^{-\mu} = -C_{10}\Phi\left(\frac{1}{2},\ \frac{3}{2};\ -\frac{\beta^2}{4a}\right) - C_{1\mu}\tau^{-\mu}\frac{\Phi\left(\mu+\frac{1}{2},\ \frac{3}{2};\ -\frac{\beta^2}{4a}\right)}{\Phi\left(\mu,\ \frac{1}{2};\ -\frac{\beta^2}{4a}\right)}, \tag{14}$$

$$\frac{Ba}{2} = C_{10}\frac{a}{\beta^2}\exp\left(-\frac{\beta^2}{4a}\right) + C_{1\mu}\tau^{-\mu}\left\{\frac{a}{\beta^2}\Phi\left(\mu+\frac{3}{2},\ \frac{3}{2};\ -\frac{\beta^2}{4a}\right) - \frac{\mu}{3}\Phi\left(\mu+\frac{3}{2},\ \frac{5}{2};\ -\frac{\beta^2}{4a}\right)\right.$$

$$\left. + \mu\frac{\Phi\left(\mu+\frac{1}{2},\ \frac{3}{2};\ -\frac{\beta^2}{4a}\right)}{\Phi\left(\mu,\ \frac{1}{2};\ -\frac{\beta^2}{4a}\right)}\Phi\left(\mu+1,\ \frac{3}{2};\ -\frac{\beta^2}{4a}\right)\right\}. \tag{15}$$

The last conditions are satisfied if

$$P = -C_{10}\Phi\left(\frac{1}{2},\ \frac{3}{2};\ -\frac{\beta^2}{4a}\right) = -\frac{C_{10}\sqrt{\pi a}}{\beta}\operatorname{erf}\left(\frac{\beta}{2\sqrt{a}}\right), \tag{16}$$

$$\frac{B}{2} = \frac{C_{10}}{\beta^2}\exp\left(-\frac{\beta^2}{4a}\right), \tag{17}$$

$$Q = -C_{1\mu}\frac{\Phi\left(\mu+\frac{1}{2},\ \frac{3}{2};\ -\frac{\beta^2}{4a}\right)}{\Phi\left(\mu,\ \frac{1}{2};\ -\frac{\beta^2}{4a}\right)} \tag{18}$$

and if μ satisfies the transcendental equation

$$\mu = f\left(\mu; \frac{\beta^2}{2a}\right), \tag{19}$$

where

$$f\left(\mu; \frac{\beta^2}{4a}\right) = \left[\frac{a}{\beta^2} \Phi\left(\mu + \frac{3}{2}, \frac{3}{2}; -\frac{\beta^2}{4a}\right) \Phi\left(\mu, \frac{1}{2}; -\frac{\beta^2}{4a}\right)\right] \Big/ \left[\frac{1}{3} \Phi\left(\mu + \frac{3}{2}, \frac{5}{2}; -\frac{\beta^2}{4a}\right) \times \Phi\left(\mu, \frac{1}{2}; -\frac{\beta^2}{4a}\right) - \Phi\left(\mu + \frac{1}{2}, \frac{3}{2}; -\frac{\beta^2}{4a}\right) \Phi\left(\mu + 1, \frac{3}{2}; -\frac{\beta^2}{4a}\right)\right]. \tag{20}$$

Equations (16) and (17) indicate that β and C_{10} will be precisely the same as in the well-known self-similar solution.

Further, assuming that a solution of the transcendental Eq. (19) exists, one can find μ and then $C_{1\mu}$. As a result, a solution can be found for the problem of (1)-(5) under the assumption $\varphi(\tau)$ satisfies the condition (13).

Numerical values of the function (20) for real arguments are given in Table 1.

The tabulated results provide a basis for considering that there are also values of μ for which the relation (19) is valid to a high degree of accuracy. For example, when $(\beta^2/4a) > 2$,

$$\left| f\left(\mu; \frac{\beta^2}{4a}\right) - \mu \right| \leqslant 10^{\mu},$$

and therefore one can always point out a value of τ for which the boundary condition (15) is satisfied with sufficient accuracy. This demonstrates that there are also other functions in addition to $\varphi(\tau)$ = const < 0 satisfying the relation (13) for which the zero isotherm grows precisely as in the self-similar solution but for which the temperature distribution in the frozen zone will be different.

NOTATION

- T is the temperature;
- τ is the time;
- a is the thermal diffusivity;
- $\xi(\tau)$ is the zeroth isotherm path;
- x is the space coordinate;
- B is the quotient from division of phase-transition enthalpy by thermal conductivity of frozen-zone material.

LITERATURE CITED

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3. I. S. Gradshtein and I. M. Ryzhik, Tables of Integrals, Sums, Series, and Products, Academic Press (1966).